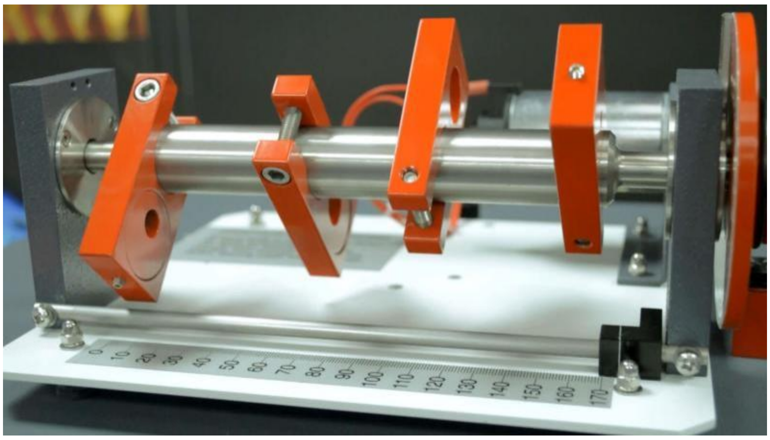
# 8

# BALANCING OF ROTATING MASSES



## *Course Contents*

**.1 Introduction**

– Often an unbalance of forces is produced in rotary or reciprocating machinery due to the inertia forces associated with the moving masses. Balancing is the process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible is eliminated entirely.

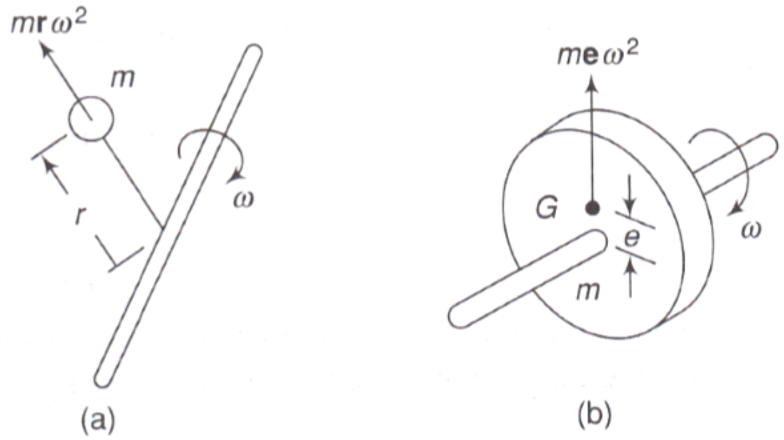



*Fig. 8.1*

– A particle or mass moving in a circular path experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force acting radially outwards acts on the axis of rotation and is known as centrifugal force [Fig. .1(a)]. This is a disturbing force on the axis of rotation, the magnitude of which is constant but the direction changes with the rotation of the mass.

– In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of the rotor lies on the axis of the shaft. When the centre of mass does not lie on the axis or there is an eccentricity, an unbalanced force is produced [Fig. .1(b)]. This type of unbalance is very common. For example, in steam turbine rotors, engine crankshafts, rotary compressors and centrifugal pumps.

– Most of the serious problems encountered in high-speed machinery are the direct result of unbalanced forces. These forces exerted on the frame by the moving machine members are time varying, impart vibratory motion to the frame and produce noise. Also, there are human discomfort and detrimental effects on the machine performance and the structural integrity of the machine foundation.

– The most common approach to balancing is by redistributing the mass which may be accomplished by addition or removal of mass from various machine members.

– There are two basic types of unbalance-rotating unbalance and reciprocating unbalance – which may occur separately or in combination.

**.2 Static Balancing:**

– A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation.

### .3 Types of Balancing:

There are main two types of balancing conditions

(i) Balancing of rotating masses

(ii) Balancing of reciprocating masses

#### (i) Balancing of Rotating Masses

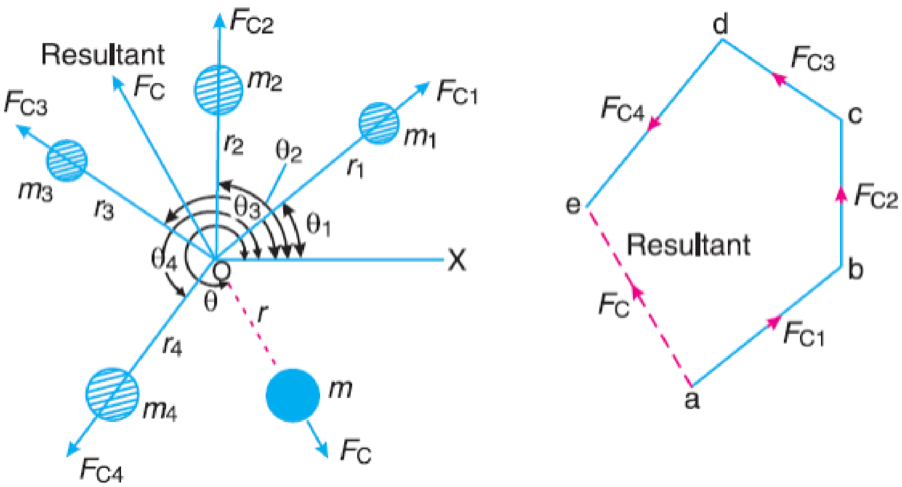
Whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal forces of both the masses are made to be equal and opposite. The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass is called balancing of rotating masses.

The following cases are important from the subject point of view:

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of different masses rotating in the same plane.
3. Balancing of different masses rotating in different planes.

### .4 Balancing of Several Masses Rotating in the Same Plane

– Consider any number of masses (say four) of magnitude $m_1$, $m_2$, $m_3$ and $m_4$ at distances of $r_1$, $r_2$, $r_3$ and $r_4$ from the axis of the rotating shaft. Let $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ be the angles of these masses with the horizontal line OX, as shown in Fig. 1.2 (a). Let these masses rotate about an axis through O and perpendicular to the plane of paper, with a constant angular velocity of $\omega$ rad/s.




*(a) Space diagram.* *(b) Vector diagram.*

*Fig. 8.2 Balancing of several masses rotating in the same plane.*

– The magnitude and position of the balancing mass may be found out analytically or graphically as discussed below:

### 1. Analytical method

– Each mass produces a centrifugal force acting radially outwards from the axis of rotation. Let F be the vector sum of these forces.

$$F = m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 + m_4r_4\omega^2$$

– The rotor is said to be statically balanced if the vector sum F is zero.

– If F is not zero, i.e., the rotor is unbalanced, then produce a counterweight (balance weight) of mass $m_c$, at radius $r_c$ to balance the rotor so that

$$m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 + m_4r_4\omega^2 + m_cr_c\omega^2 = 0$$
$$m_1r_1 + m_2r_2 + m_3r_3 + m_4r_4 + m_cr_c = 0$$

– The magnitude of either $m_c$ or $r_c$ may be selected and of other can be calculated.

– In general, if $\Sigma mr$ is the vector sum of $m_1.r_1$, $m_2.r_2$, $m_3.r_3$, $m_4.r_4$, etc., then

$$\Sigma mr + m_cr_c = 0$$

– To solve these equation by mathematically, divide each force into its $x$ and $z$ components, $\Sigma mr\cos\theta + m_cr_c\cos\theta_c = 0$

and $\Sigma mr\sin\theta + m_cr_c\sin\theta_c = 0$

$$m_cr_c\cos\theta_c = -\Sigma mr\cos\theta \qquad \text{.............................(i)}$$

and

$$m_cr_c\sin\theta_c = -\Sigma mr\sin\theta \qquad \text{...........................(ii)}$$

– Squaring and adding (i) and (ii),

$$m_cr_c = \sqrt{(\Sigma mr\cos\theta)^2 + (\Sigma mr\sin\theta)^2}$$

– Dividing (ii) by (i),

$$\tan\theta_c = \frac{-\Sigma mr\sin\theta}{-\Sigma mr\cos\theta}$$

– The signs of the numerator and denominator of this function identify the quadrant of the angle.

### 2. Graphical method

– First of all, draw the space diagram with the positions of the several masses, as shown in Fig. .2 (a).

– Find out the centrifugal force (or product of the mass and radius of rotation) exerted by each mass on the rotating shaft.

– Now draw the vector diagram with the obtained centrifugal forces (or the product of the masses and their radii of rotation), such that *ab* represents the centrifugal force exerted by the mass $m_1$ (or $m_1.r_1$) in magnitude and direction to some suitable scale. Similarly, draw *bc, cd* and *de* to represent centrifugal forces of other masses $m_2$, $m_3$ and $m_4$ (or $m_2.r_2$, $m_3.r_3$ and $m_4.r_4$).

– Now, as per polygon law of forces, the closing side *ae* represents the resultant force in magnitude and direction, as shown in Fig. .2 (b).

– The balancing force is, then, equal to resultant force, but in opposite direction.
– Now find out the magnitude of the balancing mass (m) at a given radius of rotation (r), such that

$$m.r.\omega^2 = \text{Resultant centrifugal force}$$

or $$m.r = \text{Resultant of } m_1.r_1,\ m_2.r_2,\ m_3.r_3 \text{ and } m_4.r_4$$

– (In general for graphical solution, vectors $m_1.r_1$, $m_2.r_2$, $m_3.r_3$, $m_4.r_4$, etc., are added. If they close in a loop, the system is balanced. Otherwise, the closing vector will be giving $m_c.r_c$. Its direction identifies the angular position of the counter-mass relative to the other mass.)

**Example .1** :A circular disc mounted on a shaft carries three attached masses of 4 kg, 3 kg and 2.5 kg at radial distances of 75 mm, 5 mm and 50 mm and at the angular positions of 45°, 135° and 240° respectively. The angular positions are measured counterclockwise from the reference line along the x-axis. Determine the amount of the counter-mass at a radial distance of 75 mm required for the static balance.

| | | |
|---|---|---|
| $m_1 = 4$ kg | $r_1 = 75$ mm | $\theta_1 = 45°$ |
| $m_2 = 3$ kg | $r_2 = 5$ mm | $\theta_2 = 135°$ |
| $m_3 = 2.5$ kg | $r_3 = 50$ mm | $\theta_3 = 240°$ |

$m_1r_1 = 4 \times 75 = 300$ kg.mm
$m_2r_2 = 3 \times 5 = 255$ kg.mm
$m_3r_3 = 2.5 \times 50 = 125$ kg.mm

**Analytical Method:**

$$\Sigma mr + m_cr_c = 0$$

$$300\cos45° + 255\cos135° + 125\cos240° + m_cr_c\cos\theta_c = 0 \quad \text{and}$$

$$300\sin45° + 255\sin135° + 125\sin240° + m_cr_c\sin\theta_c = 0$$

Squaring, adding and then solving,

$$m_cr_c = \sqrt{(300\cos45° + 255\cos135° + 125\cos240°)^2 + (300\sin45° + 255\sin135° + 125\sin240°)^2}$$

$$m_c \times 75 = \sqrt{(-30.\ \ )^2 + (2\ \ 4.2)^2}$$

$$= 2\ \ 5.\ \ \text{kg.mm}$$

**$m_c$ = 3. 1 kg**

$$\tan\theta_c = \frac{-\Sigma mr\sin\theta}{-\Sigma mr\cos\theta} = \frac{-2\ \ 4.2}{-(-30.\ \ )} = -9.2$$

$$\theta_c = -\ \ 3°50'$$

$\theta_c$ lies in the fourth quadrant (numerator is negative and denominator is positive).

$$\theta_c = 3\ \ 0 - \ \ 3°50'$$

$$\boldsymbol{\theta_c = 27\ \ °9'}$$

**Graphical Method:**

- The magnitude and the position of the balancing mass may also be found graphically as discussed below :
- Now draw the vector diagram with the above values, to some suitable scale, as shown in Fig.  .3. The closing side of the polygon *co* represents the resultant force. By measurement, we find that *co* = 2 5. 4 kg-mm.

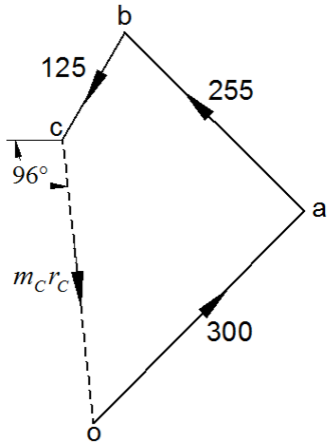



*Fig. 8.3 Vector Diagram*

- The balancing force is equal to the resultant force. Since the balancing force is proportional to *m.r*, therefore

$m_C \times 75$ = vector *co* = 2 5. 4 kg-mm or $m_C$ = 2 5. 4/75

***$m_C$* = 3. 1 kg.**

- By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal or positive X-axis,

**$\theta_C$ = 27 °.**

**Example  .2 :**Four masses $m_1$, $m_2$, $m_3$ and $m_4$ are 200 kg, 300 kg, 240 kg and 2 0 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.

| | | |
|---|---|---|
| $m_1$ = 200 kg | $r_1$ = 0.2 m | $\theta_1$ = 0° |
| $m_2$ = 300 kg | $r_2$ = 0.15 m | $\theta_2$ = 45° |
| $m_3$ = 240 kg | $r_3$ = 0.25 m | $\theta_3$ = 45° +75° = 120° |
| $m_4$ = 2 0 kg | $r_4$ = 0.3 m | $\theta_4$ = 120° + 135° = 255° |
| $m_1r_1$ = 200 x 0.2 = 40 | $r_C$ = 0.2 m | |
| $m_2r_2$ = 300 x 0.15 = 45 | | |
| $m_3r_3$ = 240 x 0.25 =  0 | | |
| $m_4r_4$ = 2 0 x 0.3 = 7 | | |

$\Sigma mr + m_c r_c = 0$

40 cos0° + 45cos45°+  0cos120° + 7 cos255° + $m_c r_c \cos\theta_c = 0$ and

40 sin 0° + 45 sin 45°+  0 sin 120° + 7  sin 255°+ $m_c r_c \sin\theta_c = 0$

Squaring, adding and then solving,

$$m_c r_c = \sqrt{\begin{matrix}(40\cos 0^\circ + 45\cos 45^\circ + \;\; 0\cos 120^\circ + \;\; 7 \;\; \cos 255^\circ)^2 + \\ (40\sin 0^\circ + 45\sin 45^\circ + \;\; 0\sin 120^\circ + \;\; 7 \;\; \sin 255^\circ)^2\end{matrix}}$$

$$m_c \times 0.2 = \sqrt{(21.\;\;)^2 + (\;\;.5)^2}$$

$$= 23.2 \text{ kg.mm}$$

$$\mathbf{m_c = 11 \;\;\; kg}$$

$$\tan\theta_c = \frac{-\sum mr\sin\theta}{-\sum mr\cos\theta} = \frac{-\;\;.5}{-21.} = 0.3935$$

$$\theta_c = 21°2\;\;'$$

$\theta_c$ lies in the third quadrant (numerator is negative and denominator is negative).

$$\theta_c = 1\;\;0 + 21°2\;\;'$$

$$\boldsymbol{\theta_c = 201°2\;\;'}$$

**Graphical Method:**

– For graphical method draw the vector diagram with the above values, to some suitable scale, as shown in Fig. .4. The closing side of the polygon *ae* represents the resultant force. By measurement, we find that *ae* = 23 kg-m.

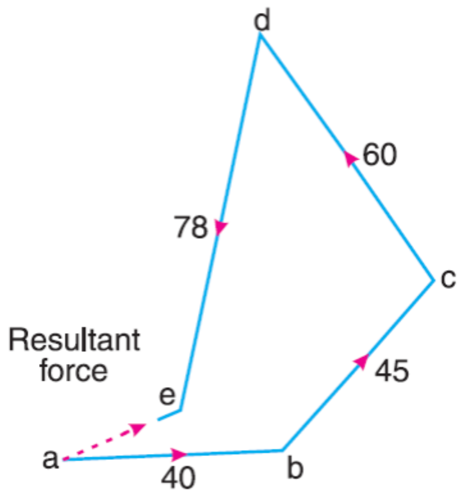



*Fig. 8.4 Vector Diagram*

– The balancing force is equal to the resultant force.Since the balancing force is proportional to *m.r*, therefore

$m \times 0.2$ = vector *ea*= 23 kg-m or $m_C = 23/0.2$

$\boldsymbol{m_C}$ **= 115 kg.**

– By measurement we also find that the angle of inclination of the balancing mass (*m*) from the horizontal or positive X-axis,

$\boldsymbol{\theta_C = 201°.}$

**.5 Dynamic Balancing**

– When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

– In the work that follows, the products of *mr* and *mrl* (instead of $mr\omega^2$ and $mrl\omega^2$), usually, have been referred as force and couple respectively as it is more convenient to draw force and couple polygons with these quantities.

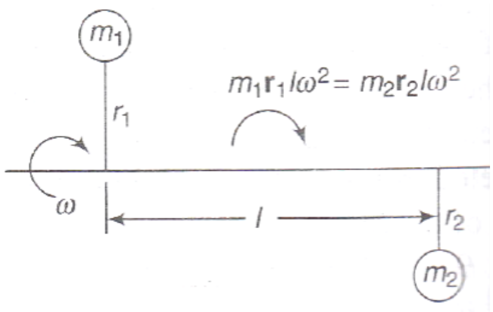



*Fig. 8.5*

– If m $_1$, and m $_2$ are two masses (Fig. .5) revolving diametrically opposite to each other in different planes such that $m_1r_1 = m_2r_2$, the centrifugal forces are balanced, but an unbalanced couple of magnitude $m_1r_1l$ (= $m_2r_2l$) is introduced. The couple acts in a plane that contains the axis of rotation and the two masses. Thus, the couple is of constant magnitude but variable direction.

**. Balancing of Several Masses Rotating in the different Planes**

– Let there be a rotor revolving with a uniform angular velocity $\omega$. $m_1$, $m_2$ and $m_3$ are the masses attached to the rotor at radii $r_1$, $r_2$ and $r_3$ respectively. The masses $m_1$, $m_2$ and $m_3$ rotate in planes1, 2 and 3 respectively. Choose a reference plane at *O* so that the distances of the planes 1, 2 and 3 from *O* are $l_1$, $l_2$ and $l_3$ respectively.

– Transference of each unbalanced force to the reference plane introduces the like number of forces and couples.

– The unbalanced forces in the reference plane are $m_1r_1\omega^2$, $m_2r_2\omega^2$ and $m_3r_3\omega^2$ acting radially outwards.

– The unbalanced couples in the reference plane are $m_1r_1\omega^2l_1$, $m_2r_2\omega^2l_2$ and $m_3r_3\omega^2l_3$ which may be represented by vectors parallel to the respective force vectors, i.e., parallel to the respective radii of $m_1$, $m_2$ and $m_3$.

– For complete balancing of the rotor, the resultant force and resultant couple both should be zero, i.e., $m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 = 0$ ....................(a)

and $m_1r_1\omega^2l_1 + m_2r_2\omega^2l_2 + m_3r_3\omega^2l_3 = 0$ ....................(b)

– If the Eqs (a) and (b) are not satisfied, then there are unbalanced forces and couples. A mass placed in the reference plane may satisfy the force equation but the couple equation is satisfied only by two equal forces in different transverse planes.

– Thus in general, two planes are needed to balance a system of rotating masses.

– Therefore, in order to satisfy Eqs (a) and (b), introduce two counter-masses $m_{C1}$ and $m_{C2}$ at radii $r_{C1}$ and $r_{C2}$ respectively. Then Eq. (a) may be written as

$$m_1r_1\omega^2 + m_2r_2\omega^2 + m_3r_3\omega^2 + m_{C1}r_{C1}\omega^2 + m_{C2}r_{C2}\omega^2 = 0$$

$$m_1r_1 + m_2r_2 + m_3r_3 + m_{C1}r_{C1} + m_{C2}r_{C2} = 0$$

$$\Sigma mr + m_{C1}r_{C1} + m_{C2}r_{C2} = 0 \qquad \text{......................(c)}$$

– Let the two countermasses be placed in transverse planes at axial locations $O$ and $Q$, i.e., the countermass $m_{C1}$ be placed in the reference plane and the distance of the plane of $m_{C2}$ be $l_{C2}$ from the reference plane. Equation (b) modifies to (taking moments about $O$)

$$m_1r_1\omega^2l_1 + m_2r_2\omega^2l_2 + m_3r_3\omega^2l_3 + m_{C2}r_{C2}\omega^2l_{C2} = 0$$

$$m_1r_1l_1 + m_2r_2l_2 + m_3r_3l_3 + m_{C2}r_{C2}l_{C2} = 0$$

$$\Sigma mrl + m_{C2}r_{C2}l_{C2} = 0 \qquad \text{.....................(d)}$$

– Thus, Eqs (c) and (d) are the necessary conditions for dynamic balancing of rotor. Again the equations can be solved mathematically or graphically.

Dividing Eq. (d) into component form

$$\Sigma mrl\cos\theta + m_{C2}r_{C2}l_{C2}\cos\theta_{C2} = 0$$

$$\Sigma mrl\sin\theta + m_{C2}r_{C2}l_{C2}\sin\theta_{C2} = 0$$

$$m_{C2}r_{C2}l_{C2}\cos\theta_{C2} = -\Sigma mrl\cos\theta \qquad \text{........................(i)}$$

$$m_{C2}r_{C2}l_{C2}\sin\theta_{C2} = -\Sigma mrl\sin\theta \qquad \text{.......................(ii)}$$

– Squaring and adding (i) and (ii)

$$m_{C2}r_{C2}l_{C2} = \sqrt{(\Sigma mrl\cos\theta)^2 + (\Sigma mrl\sin\theta)^2}$$

– Dividing (ii) by (i),

$$\tan\theta_{C2} = \frac{-\Sigma mrl\sin\theta}{-\Sigma mrl\cos\theta}$$

– After obtaining the values of $m_{C2}$ and $\theta_{C2}$ from the above equations, solve Eq. (c) by taking its components,

$$\Sigma mr\cos\theta + m_{C1}r_{C1}\cos\theta_{C1} + m_{C2}r_{C2}\cos\theta_{C2} = 0$$

$$\Sigma mr\sin\theta + m_{C1}r_{C1}\sin\theta_{C1} + m_{C2}r_{C2}\sin\theta_{C2} = 0$$

$$m_{C1}r_{C1}\cos\theta_{C1} = -(\Sigma mr\cos\theta + m_{C2}r_{C2}\cos\theta_{C2})$$

$$m_{C1}r_{C1}\sin\theta_{C1} = -(\Sigma mr\sin\theta + m_{C2}r_{C2}\sin\theta_{C2})$$

$$m_{C1}r_{C1} = \sqrt{(\Sigma mr\cos\theta + m_{C2}r_{C2}\cos\theta_{C2})^2 + (\Sigma mr\sin\theta + m_{C2}r_{C2}\sin\theta_{C2})^2}$$

$$\tan\theta_{C1} = \frac{-(\Sigma mr\sin\theta + m_{C2}r_{C2}\sin\theta_{C2})}{-(\Sigma mr\cos\theta + m_{C2}r_{C2}\cos\theta_{C2})}$$

**Example .3 :** A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 0 mm, 70 mm, 0 mm and 0 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

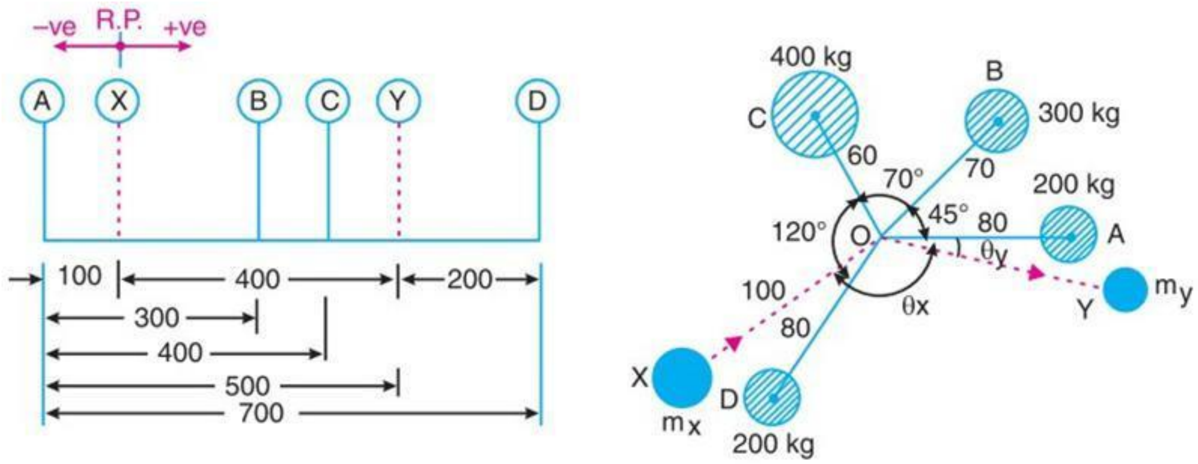
| | | | |
|---|---|---|---|
| $m_A$ = 200 kg | $r_A$ = 0 mm | $\theta_A$ = 0° | $l_A$ = -100 mm |
| $m_B$ = 300 kg | $r_B$ = 70 mm | $\theta_B$ = 45° | $l_B$ = 200 mm |
| $m_C$ = 400 kg | $r_C$ = 0 mm | $\theta_C$ = 45° +70° = 115° | $l_C$ = 300 mm |
| $m_D$ = 200 kg | $r_D$ = 0 mm | $\theta_D$ = 115° + 120° = 235° | $l_D$ = 00 mm |
| | $r_X = r_Y$ = 100 mm | | $l_Y$ = 400 mm |

Let $m_X$ = Balancing mass placed in plane *X*, and

$m_Y$ = Balancing mass placed in plane *Y*.

The position of planes and angular position of the masses (assuming the mass A as horizontal) are shown in Fig. .5 (*a*) and (*b*) respectively.

Assume the plane *X* as the reference plane (*R.P.*). The distances of the planes to the right of plane *X* are taken as + ve while the distances of the planes to the left of plane *X* are taken as –ve.




*(a) Position of planes.* *(b) Angular position of masses.*

*Fig. 8.6*

$m_A r_A l_A$ = 200 x 0.0 x (-0.1) = -1. kg.m $^2$ $m_A r_A$ = 200 x 0.0 = 1 kg.m

$m_B r_B l_B$ = 300 x 0.07 x 0.2 = 4.2 kg.m $^2$ $m_B r_B$ = 300 x 0.07 = 21 kg.m

$m_C r_C l_C$ = 400 x 0.0 x 0.3 = 7.2 kg.m $^2$ $m_C r_C$ = 400 x 0.0 = 24 kg.m

$m_D r_D l_D$ = 200 x 0.0 x 0. = 9. kg.m $^2$ $m_D r_D$ = 200 x 0.0 = 1 kg.m

**Analytical Method:**

For unbalanced couple

$\Sigma mrl + m_Y r_Y l_Y = 0$

$$m_Y r_Y l_Y = \sqrt{(\Sigma mrl\cos\theta)^2 + (\Sigma mrl\sin\theta)^2}$$

$$m_Y r_Y l_Y = \sqrt{\begin{array}{l}(-1.\ \cos 0^\circ + 4.2\cos 45^\circ + 7.2\cos 115^\circ + 9.\ \cos 235^\circ)^2 + \\ (-1.\ \sin 0^\circ + 4.2\sin 45^\circ + 7.2\sin 115^\circ + 9.\ \sin 235^\circ)^2\end{array}}$$

$$m_Y r_Y l_Y = \sqrt{(-7.179)^2 + (1.\ 3)^2}$$

$$m_Y \times 0.1 \times 0.4 = 7.3$$

**$m_Y$ = 1  4 kg.**

$$\tan\theta_Y = \frac{-\sum mrl\sin\theta}{-\sum mrl\cos\theta} = \frac{-1.\ 3}{-(-7.179)} = -0.227$$

$\theta_Y = -12°47'$

$\theta_Y$ lies in the fourth quadrant (numerator is negative and denominator is positive).

$\theta_Y$ = 3  0 –12°47'

**$\theta_Y$ = 347°12'**

For unbalanced centrifugal force

$\Sigma mr + m_X r_X + m_Y r_Y = 0$

$$m_X r_X = \sqrt{(\Sigma mr\cos\theta + m_Y r_Y\cos\theta_Y)^2 + (\Sigma mr\sin\theta + m_Y r_Y\sin\theta_Y)^2}$$

$$m_X r_X = \sqrt{\begin{array}{l}(1\ \cos 0^\circ + 21\cos 45^\circ + 24\cos 115^\circ + 1\ \cos 235^\circ + 1\ .4\cos 347^\circ 12')^2 + \\ (1\ \sin 0^\circ + 21\sin 45^\circ + 24\sin 115^\circ + 1\ \sin 235^\circ + 1\ .4\sin 347^\circ 12')^2\end{array}}$$

$$m_X r_X = \sqrt{(29.47)^2 + (19.42)^2}$$

$$m_X \times 0.1 = 35.29$$

**$m_X$ = 353  kg.**

$$\tan\theta_X = \frac{-\sum mr\sin\theta}{-\sum mr\cos\theta} = \frac{-19.42}{-29.47} = 0.\ 5\ 9$$

$\theta_X = 33°22'$

$\theta_X$ lies in the third quadrant (numerator is negative and denominator is negative).

$\theta_X$ = 1  0 +33°22'

**$\theta_X$ = 213°22'**

**Graphical Method:**

The balancing masses and their angular positions may be determined graphically as discussed below :

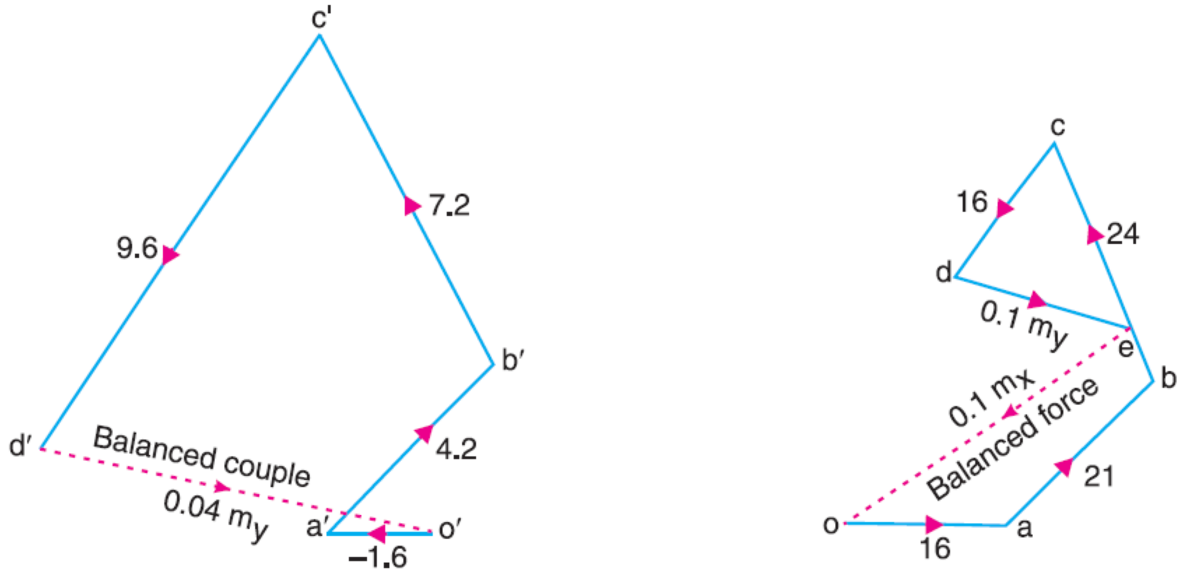
*Table 8.1*

| Plane | Angle | Mass (m) kg | Radius (r)m | Cent.force ÷ $\omega^2$ *(mr)* kg-m | Distance from Ref. Plane *(l)* m | Couple ÷ $\omega^2$ *(mrl)* kg-m$^2$ |
|---|---|---|---|---|---|---|
| A | 0° | 200 | 0.0 | 1 0 | – 0.1 | –1. |
| X (R.P.) | $\theta_X$ | $m_X$ | 0.1 | 0.1 $m_X$ | 0 | 0 |
| B | 45° | 300 | 0.07 | 21 | 0.2 | 4.2 |
| C | 115° | 400 | 0.0 | 24 | 0.3 | 7.2 |
| Y | $\theta_Y$ | $m_Y$ | 0.1 | 0.1 $m_Y$ | 0.4 | 0.04 $m_Y$ |
| D | 235° | 200 | 0.0 | 1 | 0. | 9. |

– First of all, draw the couple polygon from the data given in Table .1 (column 7) as shown in Fig. .7 (a) to some suitable scale. The vector *d o* represents the balanced couple. Since the balanced couple is proportional to 0.04 $m_Y$, therefore by measurement, 0.04$m_Y$ = vector *d o* = 73 kg-m$^2$

or **$m_Y$ = 1 2.5 kg**

– The angular position of the mass $m_Y$ is obtained by drawing $Om_Y$ in Fig. . (*b*), parallel to vector *d o*. By measurement, the angular position of $m_Y$ is $\theta_Y$ = 12° in the clockwise direction from mass $m_A$ (*i.e.* 200 kg), so **$\theta_Y$ = 3 0°– 12° = 34 °.**




*(a) Couple Polygon* *(b) Force Polygon*

*Fig. 8.7*

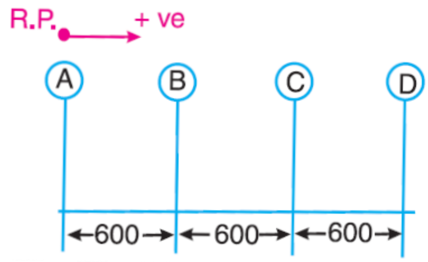
– Now draw the force polygon from the data given in Table .1 (column 5) as shown in Fig. .7 (b). The vector *eo* represents the balanced force. Since the balanced force is proportional to 0.1 $m_X$, therefore by measurement,

0.1$m_X$ = vector *eo* = 35.5 kg-m

or **$m_X$ = 355 kg.**

– The angular position of the mass $m_X$ is obtained by drawing $Om_X$ in Fig. . (*b*), parallel to vector *eo*. By measurement, the angular position of $m_X$ is $\theta_X$ = 145° in the clockwise direction from mass $m_A$ (*i.e.* 200 kg), so **$\theta_X$ = 3 0°– 145° = 215°.**

**Example .4:** Four masses A, B, C and D carried by a rotating shaft are at radii 100, 140, 210 and 1 0 mm respectively. The planes in which the masses revolve are spaced 00 mm apart and the masses of B, C and D are 1 kg, 10 kg and kg respectively. Find the required mass A and the relative angular positions of the four masses so that shaft is in complete balance.




$m_A$ = ? $r_A$ = 100mm

$m_B$ = 1 kg $r_B$ = 140mm $l_B$ = 00 mm

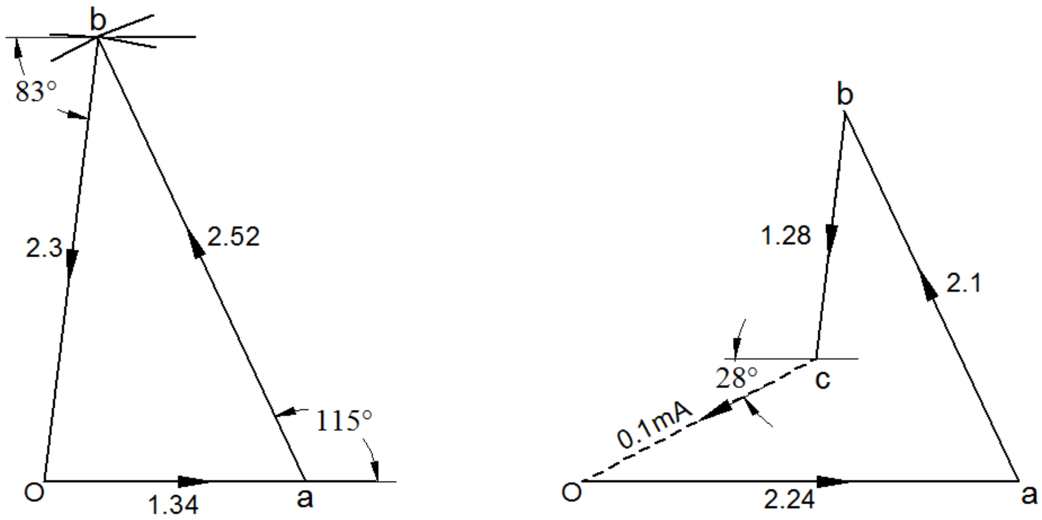
$m_C$ = 10 kg $r_C$ = 210mm $l_C$ = 1200 mm

$m_D$ = kg $r_D$ = 1 0 mm $l_D$ = 1 00 mm

*Table 8.2*

| Plane | Angle | Mass (m) kg | Radius (r) m | Cent.force ÷ $\omega^2$ *(mr)* kg-m | Distance from Ref. Plane *(l)* m | Couple ÷ $\omega^2$ *(mrl)* kg-m$^2$ |
|---|---|---|---|---|---|---|
| A (R.P.) | $\theta_A$ | $m_A$ | 0.1 | $0.1m_A$ | 0 | 0 |
| B | 0° | 1 | 0.14 | 2.24 | 0. | 1.34 |
| C | $\theta_C$ | 10 | 0.21 | 2.1 | 1.2 | 2.52 |
| D | $\theta_D$ | | 0.1 | 1.2 | 1. | 2.3 |

– First of all, draw the couple polygon from the data given in Table .2 (column 7) as shown in Fig. . (a) to some suitable scale. By measurement, the angular position of $m_C$ is $\theta_C$ **= 115°** in the anticlockwise direction from mass $m_B$ and the angular position of $m_D$ is $\theta_D$ **= 2 3°** in the anticlockwise direction from mass $m_B$.




*(a) Couple Polygon* *(b) Force Polygon*

*Fig. 8.8*

– Now draw the force polygon from the data given in Table .2 (column 5) as shown in Fig. . (b). The vector *co* represents the balanced force. Since the balanced force is proportional to 0.1 $m_A$, therefore by measurement,

$0.1m_A$ = vector *co* = 1.3 kg-m

Or **$m_A$ = 13. kg.**

– By measurement, the angular position of $m_A$ is $\theta_A$ **= 20 °** in the anticlockwise direction from mass $m_B$ (*i.e.* 1 kg).

**Example .5 :**Four masses 150 kg, 200 kg, 100 kg and 250 kg are attached to a shaft revolving at radii 150 mm, 200 mm, 100 mm and 250 mm; in planes A, B, C and D respectively. The planes B, C and D are at distances 350 mm, 500 mm and 00 mm from plane A. The masses in planes B, C and D are at an angle 105°, 200° and 300° measured anticlockwise from mass in plane A. It is required to balance the system by placing the balancing masses in the planes P and Q which are midway between the planes A and B, and between C and D respectively. If the balancing masses revolve at radius 1 0 mm, find the magnitude and angular positions of the balance masses.

$m_A$ = 150 kg  $r_A$ = 150mm  $\theta_A = 0°$

$m_B$ = 200 kg  $r_B$ = 200mm  $\theta_B = 105°$

$m_C$ = 100 kg  $r_C$ = 100mm  $\theta_C = 200°$

$m_D$ = 250 kg  $r_D$ = 250 mm  $\theta_D = 300°$

$r_X = r_Y$ = 1 0 mm

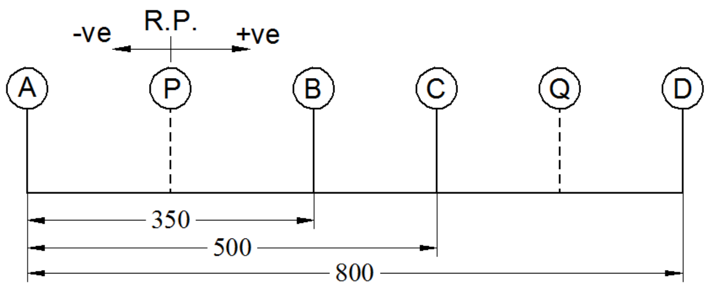



*Fig. 8.9*

*Table 8.3*

| Plane | Angle | Mass (m) kg | Radius (r) m | Cent.force ÷ $\omega^2$ *(mr)* kg-m | Distance from Ref. Plane *(l)* m | Couple ÷ $\omega^2$ *(mrl)* kg-m$^2$ |
|---|---|---|---|---|---|---|
| A (R.P.) | 0° | 150 | 0.15 | 22.5 | –0.175 | –3.94 |
| P | $\theta_P$ | $m_P$ | 0.1 | 0.1 $m_P$ | 0 | 0 |
| B | 105° | 200 | 0.2 | 40 | 0.175 | 7 |
| C | 200° | 100 | 0.1 | 10 | 0.325 | 3.25 |
| Q | $\theta_Q$ | $m_Q$ | 0.1 | 0.1 $m_Q$ | 0.475 | 0.0 55 $m_Q$ |
| D | 300° | 250 | 0.25 | 2.5 | 0. 25 | 39.0 |

**Analytical Method:**

*Table 8.4*

| *mrl*cos$\theta$ *($H_C$)* | *mrl* sin$\theta$ *($V_C$)* | *mr*cos$\theta$ *($H_F$)* | *mr* sin$\theta$ *($V_F$)* |
|---|---|---|---|
| –3.94 | 0 | 22.5 | 0 |
| 0 | 0 | 0.1 $m_P\cos\theta_P$ | 0.1 $m_P\sin\theta_P$ |
| –1. 1 | .7 | –10.35 | 3 . 4 |
| –3.05 | –1.11 | -9.4 | –3.42 |
| 0.0 55 $m_Q\cos\theta_Q$ | 0.0 55 $m_Q\sin\theta_Q$ | 0.1 $m_Q\cos\theta_Q$ | 0.1 $m_Q\sin\theta_Q$ |
| 19.53 | –33. 3 | 31.25 | –54.13 |

$\Sigma H_C = 0$

$-3.94 + 0 - 1.\ \ 1 - 3.05 + 0.0\ \ 55\ \ m_Q\cos\theta_Q + 19.53 = 0$

$0.0\ \ 55\ \ m_Q\cos\theta_Q = -10.73$

$m_Q\cos\theta_Q = -125.497$ ……………………(i)

$\Sigma V_C = 0$

$0 + 0 +\ \ .7\ \ - 1.11 + 0.0\ \ 55\ m_Q \sin\theta_Q - 33.\ \ 3 = 0$

$0.0\ \ 55\ m_Q \sin\theta_Q = 2\ \ .1$

$m_Q \sin\theta_Q = 329.59$ ………………..(ii)

$$m_Q = \sqrt{(-125.497)^2 + (329.59)^2}$$

**$m_Q$ = 352.  7 kg.**

$$\frac{m_Q \sin\theta_Q}{m_Q \cos\theta_Q} = \frac{329.59}{-125.497}$$

$\tan\theta_Q = -2.\ \ 2$

$\theta_Q = -\ \ 9.15$

$\theta_Q = 1\ \ 0 -\ \ 9.15$

**$\theta_Q$ = 110.  4°**

$\Sigma H_F = 0$

$22.5 + 0.1\ \ m_P\cos\theta_P - 10.35 - 9.4 + 0.1\ \ m_Q\cos\theta_Q + 31.25 = 0$

$22.5 + 0.1\ \ m_P\cos\theta_P - 10.35 - 9.4 + 0.1\ \ (352.\ \ 7)\cos 110.\ \ 4° + 31.25 = 0$

$0.1\ \ m_P\cos\theta_P = -11.41$

$m_P\cos\theta_P = -\ \ 3.42$

$\Sigma V_F = 0$

$0 + 0.1\ \ m_P \sin\theta_P + 3\ \ .\ \ 4 - 3.42 + 0.1\ \ m_Q \sin\theta_Q - 54.13 = 0$

$0 + 0.1\ \ m_P \sin\theta_P + 3\ \ .\ \ 4 - 3.42 + 0.1\ \ (352.\ \ 7)\sin 110.\ \ 4° - 54.13 = 0$

$0.1\ \ m_P \sin\theta_P = -40.417$

$m_P \sin\theta_P = -224.54$

$$m_P = \sqrt{(-\ \ 3.42)^2 + (-224.54)^2}$$

**$m_P$ = 233.32 kg.**

$$\frac{m_P \sin\theta_P}{m_P \cos\theta_P} = \frac{-224.54}{-\ \ 3.42}$$

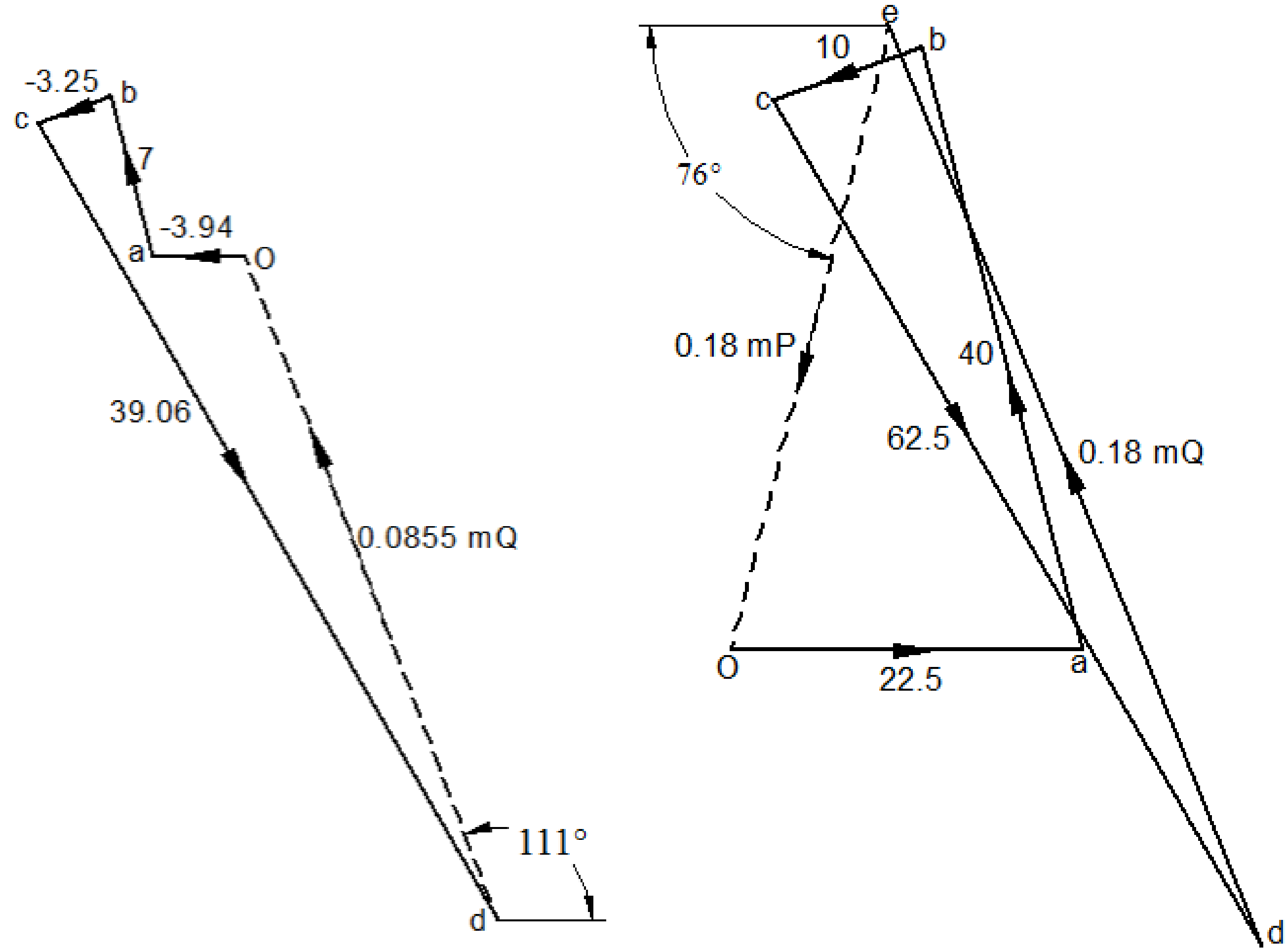
$\tan\theta_P = 3.54$

$\theta_P = 74.23$

$\theta_P = 1\ \ 0 + 74.23$

**$\theta_P$ = 254.23°**

**Graphical Method :**




*(a) Couple Polygon* *(b) Force Polygon*

*Fig. 8.10*

– First of all, draw the couple polygon from the data given in Table .4 (column 7) as shown in Fig. .10 (a) to some suitable scale. The vector *do* represents the balanced couple. Since the balanced couple is proportional to 0.0 55 $m_Q$, therefore by measurement,

$$0.0\ 55\ m_Q = \text{vector } do = 30.15\ \text{kg-m}^2$$

or **$m_Q$ = 352. 3 kg.**

– By measurement, the angular position of $m_Q$ is $\theta_Q$ = **111°** in the anticlockwise direction from mass $m_A$ (*i.e*. 150 kg).

– Now draw the force polygon from the data given in Table .4 (column 5) as shown in Fig. 1.10 (b). The vector *eo* represents the balanced force. Since the balanced force is proportional to 0.1 $m_P$, therefore by measurement,

$$0.1\ m_P = \text{vector } eo = 41.5\ \text{kg-m}$$

Or **$m_P$ = 230.5 kg.**

– By measurement, the angular position of $m_P$ is **$\theta_P$ = 25 °** in the anticlockwise direction from mass $m_A$ (*i.e*. 150kg).

**Example . :** A shaft carries four masses in parallel planes A, B, C and D in this order along its length. The masses at B and C are 1 kg and 12.5 kg respectively, and each has an eccentricity of 0 mm. The masses at A and D have an eccentricity of 0 mm. The angle between the masses at B and C is 100° and that between the masses at B and A is 190°, both being measured in the same direction. The axial distance between the planes A and B is 100 mm and that between B and C is 200 mm. If the shaft is in complete dynamic balance, determine: 1. The magnitude of the masses at A and D;

2. The distance between planes A and D; and

3. The angular position of the mass at D.

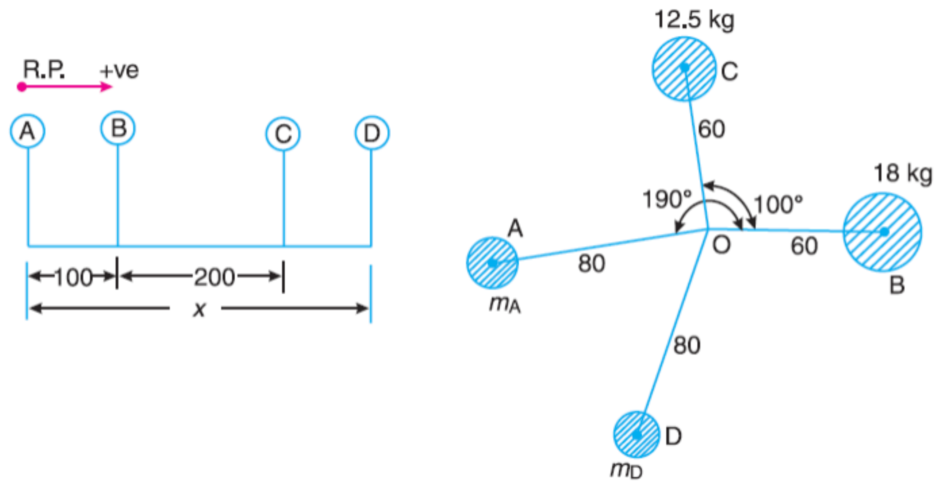
$m_A$ = ? $r_A$ = 0 mm $\theta_A$ = 190°

$m_B$ = 1 kg $r_B$ = 0 mm $\theta_B$ = 0°

$m_C$ = 12.5 kg $r_C$ = 0 mm $\theta_C$ = 100°

$m_D$ = ? $r_D$ = 0 mm $\theta_D$ = ?

*X*= Distance between planes *A* and *D*.




*(a) Position of planes.* *(b) Angular position of masses.*

*Fig. 8.11*

– The position of the planes and angular position of the masses is shown in Fig. .11 (a) and (b) respectively. The position of mass B is assumed in the horizontal direction, i.e. along OB. Taking the plane of mass A as the reference plane, the data may be tabulated as below:

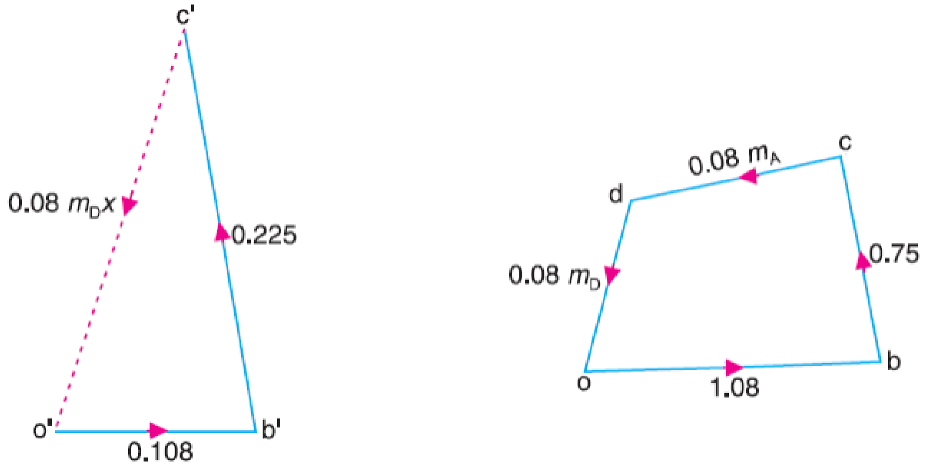
*Table 8.5*

| Plane | Angle | Mass (m) kg | Radius (r) m | Cent.force ÷ $\omega^2$ *(mr)* kg-m | Distance from Ref. Plane *(l)* m | Couple ÷ $\omega^2$ *(mrl)* kg-$m^2$ |
|---|---|---|---|---|---|---|
| A (R.P.) | 190° | $m_A$ | 0.0 | 0.0 $m_A$ | 0 | 0 |
| B | 0° | 1 | 0.0 | 1.0 | 0.1 | 0.10 |
| C | 100° | 12.5 | 0.0 | 0.75 | 0.3 | 0.225 |
| D | $\theta_D$ | $m_D$ | 0.0 | 0.0 $m_D$ | *X* | 0.0 $m_D X$ |

– First of all, draw the couple polygon from the data given in Table .5 (column 7) as shown in Fig. .12 (a) to some suitable scale. The closing side of the polygon (vector c o ) is proportional to 0.0 $m_D.X$, therefore by measurement,

$$0.0 \quad m_D X = \text{vector } c'o' = 0.235 \text{ kg-m}^2 \quad ..................(i)$$

– By measurement, the angular position of $m_D$ is $\theta_D$ = **251°** in the anticlockwise direction from mass $m_B$ (*i.e.* 1 kg).




*(a) Couple Polygon* *(b) Force Polygon*

*Fig. 8.12*

– Now draw the force polygon, to some suitable scale, as shown in Fig. .11 (b), from the data given in Table .5 (column 5), as discussed below :

i. Draw vector ob parallel to OB and equal to 1.0 kg-m.

ii. From point b, draw vector bc parallel to OC and equal to 0.75 kg-m.

iii. For the shaft to be in complete dynamic balance, the force polygon must be a closed. Therefore from point *c*, draw vector cd parallel to *OA* and from point *o* draw vector *od* parallel to *OD*. The vectors *cd* and *od* intersect at *d*. Since the vector *cd* is proportional to 0.0 $m_A$, therefore by measurement

$$0.0 \quad m_A = \text{vector } cd = 0.77 \text{ kg-m}$$

or $\mathbf{m_A = 9.\ \ 25\ kg.}$

– and vector *do* is proportional to 0.0 $m_D$, therefore by measurement,

$$0.0 \quad m_D = \text{vector do} = 0.\ \ 5 \text{ kg-m}$$

or $\mathbf{m_D = \ \ .125\ kg.}$

– Distance between planes A and D

From equation (i),

$$0.0 \quad m_D.X = 0.235 \text{ kg-m}^2$$

$$0.0 \quad \times \quad .125 \times X = 0.235 \text{ kg-m}^2$$

$$\mathbf{X = 0.3\ \ 15\ m}$$

$$\mathbf{= 3\ \ 1.5\ mm}$$

**Example .7 :** A rotating shaft carries four masses A, B, C and D which are radially attached to it. The mass centers are 30 mm, 40 mm, 35 mm and 3 mm respectively from the axis of rotation. The masses A, C and D are 7.5 kg, 5 kg and 4 kg respectively. The axial distances between the planes of rotation of A and B is 400 mm and between B and C is 500 mm. The masses A and C are at right angles to each other. Find for a complete balance,

(i) the angles between the masses B and D from mass A,

(ii) the axial distance between the planes of rotation of C and D, and

(iii) the magnitude of mass B.

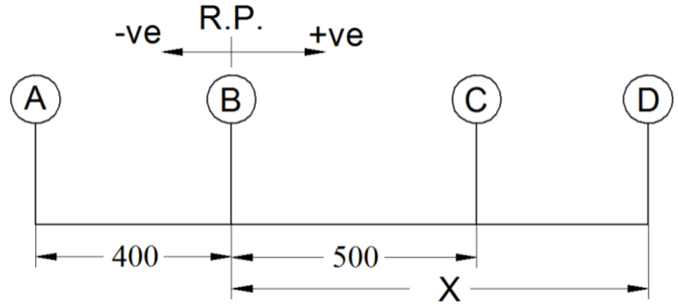
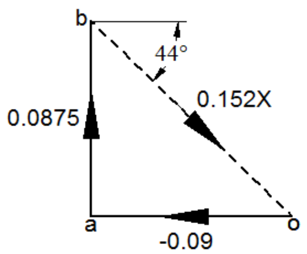
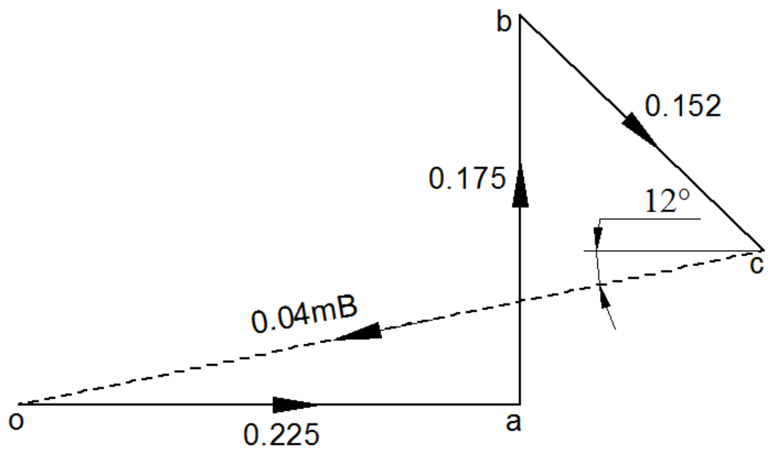



*Fig. 8.13 Position of planes*

*Table 8.6*

| Plane | Angle | Mass (m) kg | Radius (r) m | Cent.force ÷ $\omega^2$ *(mr)* kg-m | Distance from Ref. Plane *(l)* m | Couple ÷ $\omega^2$ *(mrl)* kg-m$^2$ |
|---|---|---|---|---|---|---|
| A | 0° | 7.5 | 0.03 | 0.225 | – 0.4 | –0.09 |
| B(R.P.) | $\theta_B$ | $m_B$ | 0.04 | 0.04$m_B$ | 0 | 0 |
| C | 90° | 5 | 0.035 | 0.175 | 0.5 | 0.0 75 |
| D | $\theta_D$ | 4 | 0.03 | 0.152 | *X* | 0.152*X* |




*(a) Couple Polygon* *(b) Force Polygon*

*Fig. 8.14*

– First of all, draw the couple polygon from the data given in Table 1. (column 7) as shown in Fig. .14 (a) to some suitable scale. The vector *bo* represents the balanced couple. Since the balanced couple is proportional to 0.152*X*, therefore by measurement,

$$0.152X = \text{vector } bo$$
$$= 0.13 \text{ kg-m}^2$$
or $$X = 0.\ \ 55 \text{ m.}$$

The axial distance between the planes of rotation of **C and D = 55 – 500 = 355 mm**

– By measurement, the angular position of $m_D$ is **$\theta_D$ = 3 0° – 44° = 31 °** in the anticlockwise direction from mass $m_A$ (*i.e.* 7.5 kg).

– Now draw the force polygon from the data given in Table . (column 5) as shown in Fig. .14 (b). The vector *co* represents the balanced force. Since the balanced force is proportional to 0.04 $m_B$, therefore by measurement,

$$0.04\ m_B = \text{vector } co$$
$$= 0.34 \text{ kg-m}$$
or **$m_B$ = .5 kg.**

– By measurement, the angular position of $m_B$ is **$\theta_B$ = 1 0° + 12° = 192°** in the anticlockwise direction from mass $m_A$ (*i.e.* 7.5 kg).

**Example . :** The four masses A, B, C and D revolve at equal radii are equally spaces along the shaft. The mass B is 7 kg and radii of C and D makes an angle of 90° and 240° respectively (counterclockwise) with radius of B, which is horizontal. Find the magnitude of A, C and D and angular position of A so that the system may be completely balance. Solve problem by analytically.

*Table 8.7*

| Plane | Angle | Mass (m) kg | Radius (r) m | Cent.force ÷ $\omega^2$ *(mr)* kg-m | Distance from Ref. Plane *(l)* m | Couple ÷ $\omega^2$ *(mrl)* kg-$m^2$ |
|---|---|---|---|---|---|---|
| A (R.P.) | $\theta_A$ | $m_A$ | $X$ | $m_A$ | 0 | 0 |
| B | 0° | 7 | $X$ | 7 | $Y$ | $7Y$ |
| C | 90° | $m_C$ | $X$ | $m_C$ | $2Y$ | $2m_CY$ |
| D | 240° | $m_D$ | $X$ | $m_D$ | $3Y$ | $3m_DY$ |

| *mrlcos*$\theta$ ($H_C$) | *mrl sin*$\theta$ ($V_C$) | *mrcos*$\theta$ ($H_F$) | *mr sin*$\theta$ ($V_F$) |
|---|---|---|---|
| 0 | 0 | $m_A\cos\theta_A$ | $m_A\sin\theta_A$ |
| $7Y$ | 0 | 7 | 0 |
| 0 | $2m_CY$ | 0 | $m_C$ |
| $-1.5m_DY$ | $-2.59m_DY$ | $-0.5m_D$ | $-0.\ \ m_D$ |

$$\Sigma H_C = 0$$
$$0 + 7Y + 0 - 1.5m_DY = 0$$
$$m_D = 7/1.5$$
**$m_D$ = 4. 7 kg**

$\Sigma\ V_C = 0$

$0 + 0 + 2m_C Y - 2.59m_D Y = 0$

**$m_C$ = .047 kg**

$\Sigma\ H_F = 0$

$m_A\cos\theta_A + 7 + 0 - 0.5m_D = 0$

$m_A\cos\theta_A = - 4.\quad 5$

$\Sigma\ V_F = 0$

$m_A\sin\theta_A + 0 + m_C - 0.\quad m_D = 0$

$m_A\sin\theta_A = - 2.0027$

$$m_A = \sqrt{(-4.\quad 5)^2 + (-2.0027)^2}$$

**$m_A$ = 5.07 kg**

$$\tan\theta_A = \frac{m_A \sin\theta_A}{m_A \cos\theta_A} = \frac{-2.0027}{-4.\quad 5} = 0.43$$

$\theta_A = 23.23°$

$\theta_A = 1\ \ 0° + 23.23°$

$\theta_A = 203.23°$

### 8.7 Balancing Machines

- A balancing machine is able to indicate whether a part is in balance or not and if it is not, then it measures the unbalance by indicating its magnitude and location.

#### .7.1. Static Balancing Machines

- Static balancing machines are helpful for parts of small axial dimensions such as fans, gears and impellers, etc., in which the mass lies practically in a single plane.
- There are two machine which are used as static balancing machine: Pendulum type balancing machine and Cradle type balancing machine.

**(i) Pendulum type balancing machine**

- Pendulum type balancing machine as shown in Figure .15 is a simple kind of static balancing machine. The machine is of the form of a weighing machine.
- One arm of the machine has a mandrel to support the part to be balanced and the other arm supports a suspended deadweight to make the beam approximately horizontal.
- The mandrel is then rotated slowly either by hand or by a motor. As the mandrel is rotated, the beam will oscillate depending upon the unbalance of the part.
- If the unbalance is represented by a mass m at radius r, the apparent weight is greatest when m is at the position I and least when it is at B as the lengths of the arms in the two cases will be maximum and minimum.

- A calibrated scale along with the pointer can also be used to measure the amount of unbalance. Obviously, the pointer remains stationary in case the body is statically balanced.

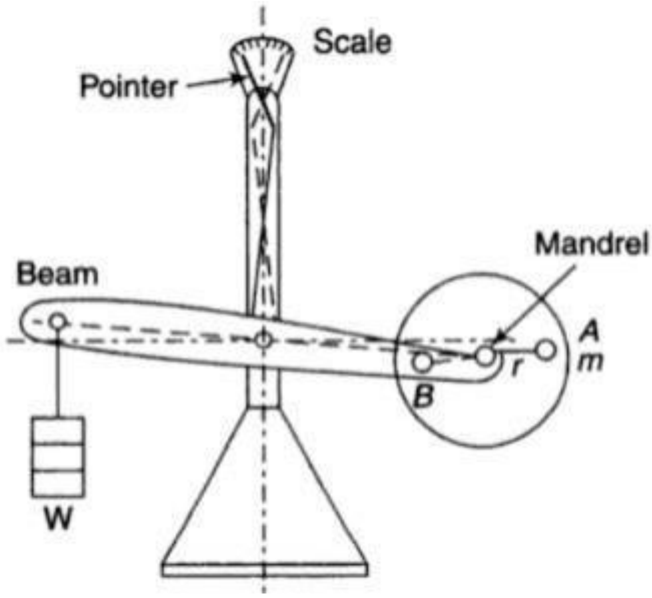



Fig. .15

**(ii) Cradle type balancing machine**

- Cradle type balancing machine as shown in fig. 1.1 is more sensitive machine than the pendulum type balancing machine.
- It consists of a cradle supported on two pivots P-P parallel to the axis of rotation of the part and held in position by two springs S-S.
- The part to be tested is mounted on the cradle and is flexibly coupled to an electric motor. The motor is started and the speed of rotation is adjusted so that it coincides with the natural frequency of the system.
- Thus, the condition of resonance is obtained under which even a small amount of unbalance generates large amplitude of the cradle.
- The moment due to unbalance = $(mr\omega^2 \cos\theta).l$ where $\omega$ is the angular velocity of rotation. Its maximum value is $mr\omega^2 l$. If the part is in static balance but dynamic unbalance, no oscillation of the cradle will be there as the pivots are parallel to the axis of rotation.

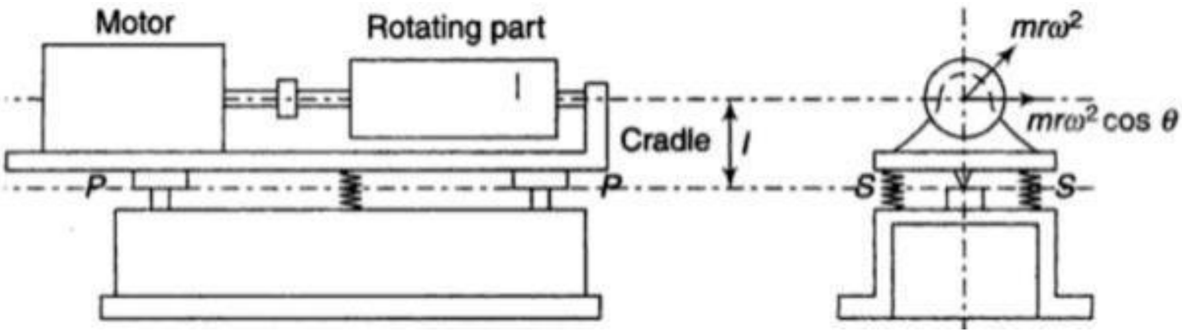



Fig. .1

### .7.2. Dynamic Balancing Machines

- For dynamic balancing of a rotor, two balancing or counter-masses are required to be used in any two convenient planes. This implies that the complete unbalance of any rotor system can be represented by two unbalances in those two planes.
- Balancing is achieved by addition or removal of masses in these two planes, whichever is convenient. The following is a common type of dynamic balancing machine.

**Pivoted-cradle Balancing Machine**

- Fig .17 shows a pivot cradle type dynamic balancing machine. Here, part which is required to be balanced is to be mounted on cradle supported by supported rollers and it is connected to drive motor through universal coupling.
- Two planes are selected for dynamic balancing as shown in fig. .17 where pivots are provided about which the cradle is allowed to oscillate.
- As shown in fig .17, right pivot is released condition and left pivot is in locked position so as to allow the cradle and part to oscillate about the pivot.
- At the both ends of the cradle, the spring and dampers are attached such that the natural frequency can be adjusted and made equal to the motor speed. Two amplitude indicators are attached at each end of the cradle.
- The permanent magnet is mounted on the cradle which moves relative to stationary coil and generates a voltage which is directly proportional to the unbalanced couple. This voltage is amplified and read from the calibrated voltmeter and gives output in terms of kg-m.
- When left pivot is locked, the unbalanced in the right correction plane will cause vibration whose amplitude is measured by the right amplitude indicator.
- After that right pivot is locked and another set of measurement is made for left hand correction plane using the amplitude indicator of the left hand side.

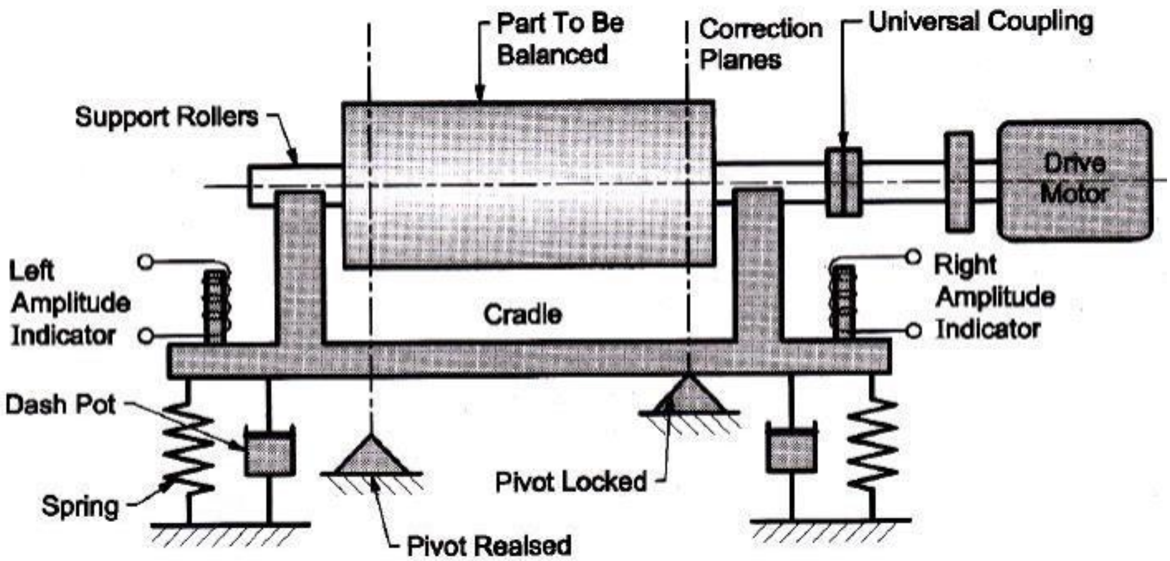



Fig. .17